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FULLY DEVELOPED FRICTIONAL AND HEAT-TRANSFER CHARACTERISTICS OF LAMINAR FLOW IN POROUS TUBES

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Abstract—The frictional and heat-transfer characteristics of fully developed laminar flow in porous tubes have been examined utilizing numerical solutions to the flow and energy equations. Based on the assumption of a constant-property fluid and uniform wall mass transfer, universal wall-friction and heat-transfer results for flow in constant temperature tubes are obtained. These are presented in the form of friction-coefficient and Nusselt-number curves spanning the continuous range of injection and extraction wall Reynolds numbers for which fully developed conditions exist.

NOMENCLATURE

- A, dimensionless axial pressure gradient, equation (7);
- C_F, wall friction coefficient, $2\tau_w/\rho\bar{u}^2$;
- C_p, specific heat;
- d, tube diameter;
- f, dimensionless stream function;
- F, dimensionless axial velocity function, $u/\bar{u}$;
- G, dimensionless radial velocity function, v/v_w;
- h, heat-transfer coefficient;
- H, dimensionless temperature function, equation (14);
- k, thermal conductivity;
- Nu, Nusselt number;
- P, static pressure;
- Pr, Prandtl number;
- q_w, wall heat flux;
- r, radial coordinates;
- Re, axial Reynolds number, $\bar{u}d/\nu$;
- Re_w, wall Reynolds number, $v_w d/\nu$;
- T, static temperature;
- u, axial velocity;
- $\bar{u}$, axial velocity averaged over tube cross-section;
- v, radial velocity;
- x, axial coordinate.

Greek symbols

- α, thermal diffusivity;
- β, momentum flux factor;
- ζ, dimensionless radial coordinate, r/r_w;
- η, dimensionless radial coordinate, ζ^2;
- λ, eigenvalue;
- ν, kinematic viscosity;
- ξ, dimensionless axial coordinate, x/r_w;
- ρ, density;
- τ, tangential shear stress.

Subscripts

- b, bulk fluid conditions;
- w, wall conditions.

INTRODUCTION

IT HAS previously been shown that the uniform injection or extraction of fluid through the walls of porous tubes can have an appreciable effect on the dynamic characteristics of fully developed

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laminar flows. This work is well summarized in the paper by Weissberg [1], particular reference being made here to the contributions of Yuan and Finkelstein [2], Berman [3], and Eckert *et al.* [4]. Related studies of the characteristics of laminar flow in two-dimensional porous channels may also be of interest. A listing of these can be found in the note by Terrill [5].

In considering these prior works, it is noted that the frictional aspects of the flow have received incomplete treatment. In particular, it was observed, qualitatively, nearly a decade ago that the wall friction is greatly reduced by fluid extraction at the tube wall [3]. Indeed, it was found to vanish at some critical mass extraction rate. However, details pertinent to the calculation of the wall friction have never been given. Such information is directly applicable to a variety of engineering problems, for example, the study of film condensation in vertical tubes. This becomes immediately obvious when it is realized that in many cases the vapor–liquid interface separating the core-flow (vapor) and annular-film (condensate) may be treated as a stationary permeable boundary (i.e. porous wall). Recent work [6] has emphasized the importance of fluid forces at the interface, including interfacial shear, on determining the flow behavior of the film.

A somewhat similar situation exists concerning previous treatments of the companion effects of surface mass transfer on the heat-transfer characteristics of laminar tube-flows. In this regard, mention is made of the several contributions made by Yuan and co-workers [7–9]. In considering each of these works, it is noted that attention was given only to the case of fluid injection at the tube wall. Furthermore, in obtaining solutions to the appropriate conservation equations, a perturbation technique was employed throughout. Thus the results obtained show only the first order departures from the classical situation of flow in an impermeable tube. Up to the present time, results applicable to general wall injection as well as wall extraction rates have not been reported, even for the case of constant property flows.

In the present paper, the effect of wall mass transfer on the fully developed frictional and heat-transfer characteristics of constant-property laminar flows is examined. Based on numerical solutions to the tube-flow and energy equations, universal friction and heat-transfer results are obtained over the continuous range of wall injection and extraction rates for which fully developed conditions exist. Wherever possible, comparisons are made with existing results based on perturbation solutions from previous investigations.

THE VELOCITY PROBLEM

Consideration is given to a constant-property fluid flowing in a straight tube of circular cross-section, at the walls of which there is uniform mass transfer. If the tube axis is taken to lie along the positive x-axis, the appropriate equations describing the fluid motion are

$$\frac{1}{r}\frac{\partial}{\partial r}(ruv) + \frac{\partial}{\partial x}(u^2) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial x^2}\right] \tag{1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rv^2) + \frac{\partial}{\partial x}(uv) = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)\right) + \frac{\partial^2 v}{\partial x^2}\right] \tag{2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{\partial u}{\partial x} = 0. \tag{3}$$

Fully developed velocity profiles

Yuan and Finkelstein [2] have shown that the foregoing equations can be reduced to ordinary differential form by the introduction of a dimensionless stream function, f. In terms of this quantity, the fully developed axial and radial velocities become

$$u = 2\bar{u}f'(\eta) \tag{4}$$

$$v = -2v_w\frac{f(\eta)}{\eta^{1/2}}. \tag{5}$$

With these and the expression of overall mass conservation, $d\bar{u}/dx = 2v_w/r_w$, the continuity equation is satisfied identically, and the axial momentum equation, equation (1), reduces to

$$\eta f''' + f'' - \frac{Re_w}{2}(f'^2 - ff'') + \frac{A}{16} = 0 \qquad (6)$$

where A is the dimensionless axial pressure gradient, a constant, given by

$$A = -\frac{Re}{\rho\bar{u}^2}\frac{\partial P}{\partial \xi}. \qquad (7)$$

In fully developed flow, the radial momentum equation no longer enters into the determination of the flow field and is only required if the radial static pressure distribution is to be determined.

Many investigators have obtained solutions to equation (6) using a variety of techniques. These works are listed in [1] and need not be discussed here. It is only noted that four boundary conditions on the velocity distributions are required in the solution of this third order equation and the determination of the unknown constant A. These conditions may be stated in terms of the velocity variables as follows: at $r = 0$, $v = 0$ and $\partial u/\partial r = 0$; at $r = r_w$, $u = 0$ and $v = -v_w$. In the present work, solutions to equation (6) were obtained numerically for parametric values of the wall Reynolds numbers, Re_w (considered positive for injection and negative for extraction).

Fully developed wall friction

The dimensionless axial pressure gradient, an unknown for the problem and denoted by A in equation (6), contains wall-friction effects as well as those due to changes in axial momentum flux. This can be clearly demonstrated by integrating equation (6) across the entire tube cross section. Making use of the appropriate boundary conditions, it follows that

$$A = -16f''(1) + 16Re_w \int_0^1 f'^2\, d\eta. \qquad (8)$$

It may be verified that the term $f''(1)$ in the foregoing equation is related to the wall shear stress coefficient and axial flow Reynolds number by the expression

$$f''(1) = \frac{-Re}{8}\left(\frac{\tau_w}{\rho\bar{u}^2}\right) = -\frac{C_F Re}{16}. \qquad (9)$$

In addition, the integral appearing in equation (8) can be written

$$\int_0^1 f'^2\, d\eta = \frac{1}{2(r_w\bar{u})^2}\int_0^{r_w} u^2 r\, dr = \frac{\overline{u^2}}{4(\bar{u})^2}. \qquad (10)$$

A momentum flux factor may now be defined such that

$$\beta = \frac{\overline{u^2}}{(\bar{u})^2}. \qquad (11)$$

With this notation, equation (8) becomes

$$A = C_F Re + 4\beta Re_w. \qquad (12)$$

From the foregoing, it is seen that the wall friction may be calculated once the pressure gradient term A is determined and the momentum flux factor β is computed. Alternatively, the wall friction can also be calculated from equation (9) once the quantity $f''(1)$ is known. In any event, a different value of A and β, or $f''(1)$, will be obtained for each chosen value of Re_w. Since this latter quantity is the only parameter appearing in the problem, it follows that the $C_F Re$ product is only a function of Re_w. This functional relationship may be termed the universal law of wall friction, results for which will be presented in a later section.

THE TEMPERATURE PROBLEM

The temperature distribution within the fluid is determined from the energy conservation equation which can be written

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right). \qquad (13)$$

In the foregoing, the fluid properties are considered to be constant and axial heat conduction has been neglected relative to radial heat conduction.

Fully developed temperature profiles

Attention is now focused on the thermally fully developed regime within the fluid. Specifying that the tube wall temperature be maintained at a uniform value, temperature profiles within the fluid are sought which have the following self-similar form

$$\frac{T - T_w}{T_b - T_w} = H(\zeta). \quad (14)$$

Introducing the foregoing into the energy equation along with the fully developed velocity functions given by $F = u/\bar{u}$ and $G = -v/v_w$, one obtains

$$\left[\frac{Re\,Pr\,r_w}{2(T_b - T_w)}\frac{\mathrm{d}T_b}{\mathrm{d}x}\right]FH - \frac{Re_w\,Pr}{2}G\frac{\mathrm{d}H}{\mathrm{d}\zeta} = \frac{1}{\zeta}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\zeta\frac{\mathrm{d}H}{\mathrm{d}\zeta}\right). \quad (15)$$

From an inspection of equation (15), it can be seen that solutions of the assumed form are only possible if the coefficient of the first term on the left-hand side is a constant, say $-\lambda^2$. That is

$$\frac{Re\,Pr\,r_w}{2(T_b - T_w)}\frac{\mathrm{d}T_b}{\mathrm{d}x} = -\lambda^2. \quad (16)$$

Making this substitution, the energy equation becomes

$$\lambda^2 FH + \frac{Re_w\,Pr}{2}GH' + \frac{1}{\zeta}(\zeta H')' = 0 \quad (17)$$

where the prime denotes differentiation with respect to ζ. The condition of uniform wall temperature and temperature-profile symmetry require that the solution to equation (17) satisfy the following boundary conditions

$$H(1) = 0;\ H'(0) = 0. \quad (18)$$

The homogeneity of the differential equation and associated boundary conditions, equations (17) and (18), identify this as an eigenvalue problem. The constant λ is the eigenvalue and the function H is the corresponding eigenfunction.

It is known that an infinite but discrete set of eigenvalues and associated eigenfunctions can in general be found to satisfy the foregoing mathematical system. However, it may be shown that the thermally fully developed temperature profile corresponds to the first eigenfunction. Thus, it is sufficient for the purposes of the present paper to obtain solutions to equation (17) corresponding to the first eigenvalue, λ_1. These have been obtained using standard numerical integration techniques for parametric values of Re_w and Pr.

The fully developed Nusselt number

The fully developed Nusselt number may now be specified with the aid of an energy balance applied to a ring-shaped element of fluid bounded by the tube walls. Thus, the wall heat flux, q_w, is given by

$$-q_w = \rho\bar{u}C_p\frac{r_w}{2}\frac{\mathrm{d}T_b}{\mathrm{d}x} + \rho C_p v_w(T_b - T_w) \quad (19)$$

where it is assumed that the injected flow ($v_w > 0$) enters the stream at the wall temperature, T_w. Also, q_w is considered positive if heat is transferred from the fluid to the tube wall.

It is customary to represent heat-transfer results in terms of a heat-transfer coefficient, $h = q_w/(T_b - T_w)$, and a Nusselt number, hd/k. Therefore, it follows from equation (19) that

$$Nu = \frac{hd}{k} = -\frac{Re\,Pr\,r_w}{2(T_b - T_w)}\frac{\mathrm{d}T_b}{\mathrm{d}x} - Re_w\,Pr. \quad (20)$$

Making use of equation (16), the Nusselt number may be written in the following condensed form

$$Nu = \lambda_1^2 - Re_w\,Pr. \quad (21)$$

It is thus seen that the fully developed Nusselt number is completely specified by the first eigenvalue and the parameters Re_w and Pr.

RESULTS AND DISCUSSION

One of the results of primary importance to the present investigation is the functional relationship between the $C_F Re$ product and the injection and extraction wall Reynolds numbers. This relationship may be termed the universal law of friction. It is presented in Fig. 1 along with the results obtained from the perturbation solution of [2].

It may be seen from Fig. 1 that about the point of zero wall Reynolds number, the results obtained from the perturbation solution are essentially indistinguishable from those obtained in the present calculations. In terms of the present notation, the former result can be written

$$C_F Re = 16\left[1 + \frac{Re_w}{24} - \frac{13}{2160}(Re_w)^2\right]. \quad (22)$$

The foregoing is seen to be a reasonable approximation to the exact calculations for $|Re_w| < 2$. Beyond this range, the departure between the exact and approximate results becomes signficant.

For large values of the injection wall Reynolds number, the values of $C_F Re$ calculated in the present work approach the asymptotic value deduced from [2] and given by $C_F Re = 19{\cdot}739$. The upper dashed-curve shown in Fig. 1 represents the asymptotic solution for large Re_w also given in [2]. This may be written

$$C_F Re = 16\left(1{\cdot}2337 + \frac{0{\cdot}3708}{Re_w}\right). \quad (23)$$

It can be seen that this result approaches the asymptote from above; therefore, it is not useful for approximating the exact results for finite values of Re_w.

Turning now to the case of fluid extraction, it is seen from Fig. 1 that the wall friction decreases monotonically with increasing extraction wall Reynolds numbers. Furthermore, there exists a definite value for Re_w at which the wall friction is reduced to zero. This behavior was previously noted qualitatively in [3]. Although it was not practical to carry the calculations to the precise limit of vanishing wall shear, due to the slow convergence of the numerical procedure near that exact point, a reasonable extrapolation of the present results places the limiting value of Re_w at no less than $-4{\cdot}626$. The last calculation was carried out for $Re_w = -4{\cdot}618$. The prior work of Berman [3] established this limit at

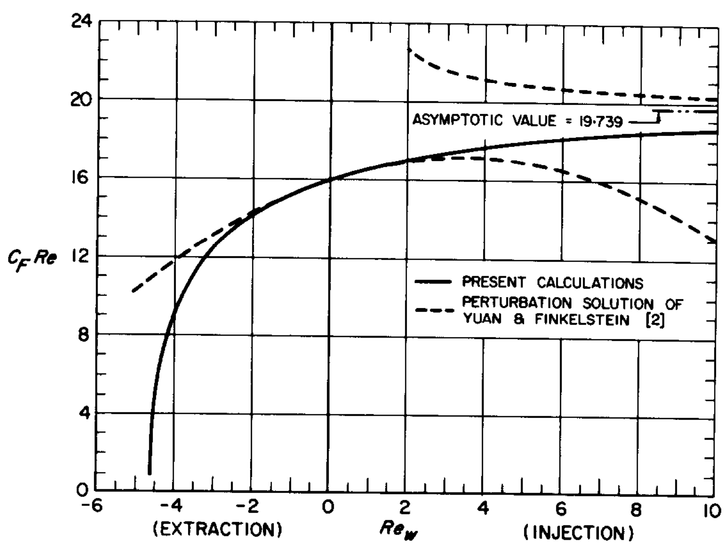

FIG. 1. Universal law of wall friction for fully developed laminar flow in porous tubes.

approximately 4·6. Comments on the physical significance of this precise limit can be found in the work by Weissberg [1].

The momentum flux parameter, β, defined by equation (11) is presented in Fig. 2. Taking cognizance of the fact that this quantity assumes the value of unity for slug flow (i.e. when $u/\bar{u} \equiv 1$) it provides an indication of the relative flatness of the velocity profiles. It is interesting to note, however, that the asymptotic limit on β as $Re_w \to \infty$ does not correspond to a slug flow condition. This limit can be calculated from the results of [2] and corresponds to $\beta = 1{\cdot}2337$. This is also denoted in Fig. 2. The present calculations approach this asymptotic limit as required.

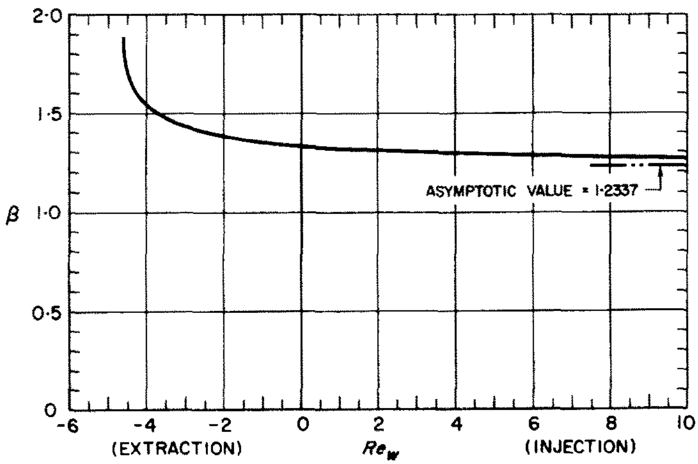

FIG. 2. Variation of momentum flux factor with wall Reynolds number.

The results of the foregoing figure plus those of Fig. 1 can now be combined, according to equation (12), to give the dimensionless axial pressure gradient. This result will not be given here. It has previously been presented in [3], wherein the symbol K was used to denote the pressure gradient rather than the symbol A. It is only noted that for sufficiently large injection wall Reynolds number, the dimensionless axial pressure gradient approaches (from below) the asymptotic linear relationship deduced from the results of [2] and given by

$$-\frac{Re}{\rho\bar{u}^2}\frac{\partial P}{\partial \xi} = 21{\cdot}205 + 4{\cdot}9348\, Re_w. \qquad (24)$$

The calculated values of the pressure gradient obtained from the present calculations are within 10 per cent of those obtained from equation (24) for $Re_w \geqslant 3$.

The results of the present heat-transfer calculations are summarized in Figs. 3 and 4. In the first of these figures typical temperature profiles are displayed; the latter figure presents the fully developed Nusselt number results.

The temperature profiles shown in Fig. 3 were computed for a Prandtl number of unity. Other values of this parameter could have been chosen for purposes of this illustration without altering the important features of these curves. Of particular interest is the fact that fluid extraction does not have a very pronounced effect on the temperature profiles. This is in sharp contrast to the behavior already found concerning the effect of fluid extraction on the velocity profiles

[3]. To emphasize this difference, a suction-modified velocity profile is shown as the dashed curve in Fig. 3. It can be seen that it exhibits some similarity to the injection-modified temperature profile. If the corresponding velocity profile for $Re_w = 8$ were plotted, it would lie nearly on the temperature curve for $Re_w = 0$.

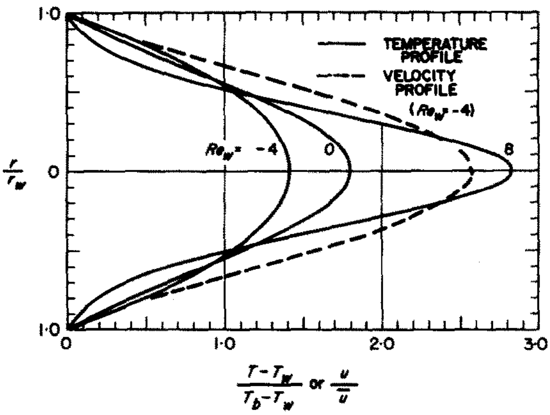

FIG. 3. Representative radial temperature profiles for $Pr = 1{\cdot}0$.

The effect of wall injection and extraction on the fully developed Nusselt number is illustrated in Fig. 4. Before discussing these results, however, it is to be noted that the classical fully developed Graetz problem for flow in an impermeable-wall tube ($Re_w = 0$) is included as a special case of the present formulation. For this case the fully developed Nusselt number is known to be independent of Prandtl number and in the present calculations was found to have the value $Nu = 3{\cdot}6568$. This value is in exact agreement with that calculated in [10] and agrees closely with Graetz's original value of 3·656.

Referring to Fig. 4, it can be seen that when there is mass injection or extraction at the tube wall the Nusselt number is strongly affected by the fluid Prandtl number. This can be traced to the fact that the quantity of heat injected into or extracted from the main stream with the wall fluid depends greatly on the fluid heat capacity, a quantity which is used in forming the Prandtl modulus. This follows from the heat balance

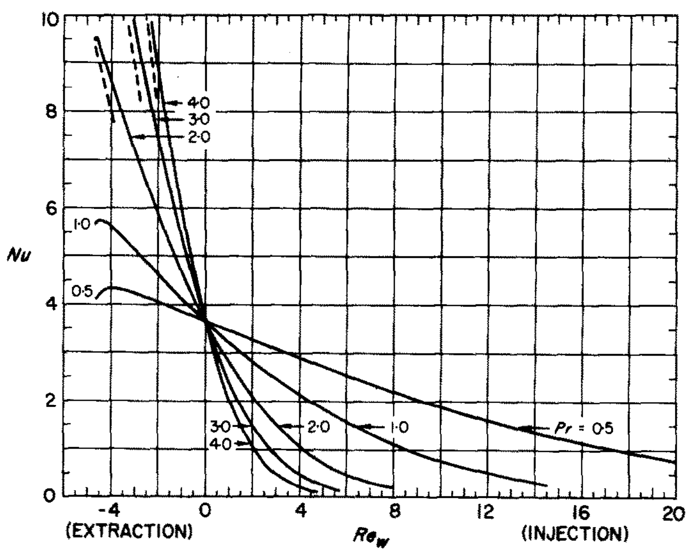

FIG. 4. Fully developed Nusselt number for flow in constant temperature porous tubes.

already applied to the fluid and given by equation (19).

For a fixed Prandtl number fluid, the effect of mass injection is to reduce the Nusselt number and thus the wall heat transfer for a given wall-to-bulk temperature difference. In the case of fluid extraction, the opposite effect occurs, and the Nusselt number increases with increasing extraction wall Reynolds number until the point of vanishing wall shear stress is approached. It will be recalled that this occurs at approximately $Re_w = -4{\cdot}62$. In the vicinity of this point, the Nusselt number turns sharply down towards zero. This occurs for all Prandtl numbers but is most easily discernible in Fig. 4 for the curves corresponding to $Pr = 0{\cdot}5$ and $1{\cdot}0$.

For large Prandtl number fluids and sufficiently large extraction wall Reynolds numbers, it was found that the first eigenvalue for the problem becomes vanishingly small. This may be traced to the fact that under these conditions the radial convective heat transport is large relative to the axial convective heat transport [see equation (17)]. It thus follows from equation (21) that $Nu \to -Re_w Pr$. This behavior is illustrated graphically in Fig. 4. It can be seen from the figure that the Nusselt number curves for $Pr \geqslant 2{\cdot}0$ approach the dashed asymptotes given by the parametric equation $Nu = -Re_w Pr$.

CONCLUDING REMARKS

Utilizing numerical solutions to the fully developed tube-flow and energy equations, it has been verified that for a given axial flow Reynolds number the effect of fluid injection is to increase the wall friction and to decrease the wall heat transfer. This behavior is in contradistinction to that found for flat-plate boundary-layer flows, for which case both the wall friction and heat transfer are reduced by fluid injection normal to the wall.

The phenomenological differences between these two flows can be traced to the effect of the axial pressure gradient on the flow dynamics. For example, in the case of boundary-layer flow over a flat plate at zero incidence, the pressure gradient in the flow direction is zero, even in the presence of uniform injection normal to the plate surface. The situation is altogether different in the case of laminar tube flow with fluid injection normal to the wall. In this case, the axial pressure gradient is not zero, but is determined by the internal flow dynamics. Furthermore, the pressure gradient is a function of wall injection rate. Because the wall friction is intimately connected with the pressure gradient as well as changes in the momentum flux in the flow direction, it is not to be unexpected that the frictional characteristics of laminar boundary-layer and fully developed internal flows would be quite different.

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Résumé—Les caractéristiques de frottement et de transport de chaleur de l'écoulement laminaire entièrement établi dans des tubes poreux ont été examinées en utilisant des solutions numériques des équations de l'écoulement et de l'énergie. On a obtenu des résultats universels pour le frottement à la paroi et le transport de chaleur avec un écoulement dans des tubes à température constante en se basant sur l'hypothèse d'un fluide à propriétés constantes et d'un transport de masse uniforme à la paroi. Ils sont présentés sous la forme de courbes de coefficient de frottement et de nombres de Nusselt s'étendant sur une gamme continue de nombres de Reynolds d'injection et d'aspiration pariétales pour lesquels il existe des conditions entièrement développées.

Zusammenfassung—Die Reibungs- und Wärmeübergangscharakteristik einer voll ausgebildeten laminaren Strömung in porösen Rohren wurde mit Hilfe von numerischen Lösungen der Strömungs- und Energiegleichung überprüft. Unter der Annahme konstanter Flüssigkeitseigenschaften und einheitlichem Stoffübergang an der Wand wurden universelle Ergebnisse für Wandreibung und Wärmeübergang bei Strömung in Rohren konstanter Temperatur erhalten. Sie sind wiedergegeben in der Form von Kurven für Reibungskoeffizienten und Nusselt-Zahlen, die den kontinuierlichen Bereich von Nusselt-Zahle für Einblasen und Absaugen überspannen, in dem voll ausgebildete Bedingungen vorliegen.

Аннотация—С помощью численных решений уравнений импульса и энергии определяются характеристики процессов трения и теплообмена при полностью развитом ламинарном течении в трубах с пористыми стенками. Получены универсальные результаты по трению на стенке и теплообмену течении в трубах для случая постоянной температуры при условии, что свойства жидкости постоянны и имеет место однородный массообмен на стенке. Результаты представлены в виде кривых изменения коэффициента трения и числа Нуссельта при постоянных числах Рейнольдса для вдува и отсоса на стенке в условиях полностью развитого течения.